UDC 533.6.011

INVESTIGATION OF A THREE-DIMENSIONAL HYPERSONIC VISCOUS SHOCK LAYER IN THE STAGNATION POINT NEIGHBORHOOD IN THE PRESENCE OF INJECTION OR SUCTION

PMM Vol.43, No.5, 1979, pp. 817-828
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(Received September 4, 1978)

Three-dimensional flow over smooth blunted bodies of a hypersonic stream of a homogeneous viscous gas in the presence of injection or suction is considered. A numerical solution is obtained for the problem in a wide range of Reynolds numbers and of injection (suction) parameters. Velocity and temperature profiles across the shock layer are presented, as well as the dependence of the coefficients of friction and heat transfer at the body surface, on the Reynolds number and the injection (suction) parameter; the dependence on the injection parameter is universal. An approximate analytic solution of the problem, which is in satisfactory agreement with the numerical solution for low Reynolds numbers is obtained using the integral method of successive approximations.

The asymptotics of equations of the hypersonic shock layer is analyzed in the case of high Reynolds numbers in the presence of injection. An analytic solution is presented of the problem of strong injection in the approximation of two inviscid layers separated by a contact surface. Solution of the boundary layer equations is obtained using the same difference grid, and it is shown that the distinction from solutions of the shock layer in terms of the coefficient of friction in the direction of the body cross section with the largest curvature radius can be considerable even at Reynolds numbers of the order of 10^8. This distinction increases with increasing suction and decreases with increase of injection. With a fairly large injection parameter the friction coefficient calculated using boundary layer equations is the same as the coefficient obtained from solutions of shock layer equations at high Reynolds numbers.

Certain results of numerical solution of equations of the mixing layer which forms close to the contact surface with considerable injection from the body surface are presented.

The effect of viscosity and thermal conductivity behind a considerably curved shock wave was first considered in [1]. Further investigation of plane and axisymmetric flows of rarified gases, using the theory of hypersonic viscous shock layer with generalized Rankin — Hugoniot relations behind the shock wave [2] were carried out by a number of authors (see [3 —10] and other). Three-dimensional hypersonic flow of viscous heat-conducting gas past blunt bodies was calculated in [11, 12].

1. Equations of three-dimensional hypersonic viscous boundary layer. Boundary conditions. Let us consider the three-dimensional flow over a smoothly blunted body. We introduce some arbitrary curvilinear coordinate system normally attached to the body surface.

Let $x^3 = \text{const}$ be the equations of a set of surfaces parallel to the body surface $x^3 = 0$ and x^1 and x^2 selected on the surface. The equations of the three-dimensional hypersonic viscous shock layer in a homogeneous gas in the system of coordinates $\{x^i\}$ are of the form [12]

$$\frac{\partial}{\partial x^i}\left(\rho u[i]\sqrt{\frac{a}{a_{(ii)}}}\right)=0 \tag{1.1}$$

$$\rho\left(Du[\gamma]+A^{\gamma}_{\alpha\beta}u[\alpha]u[\beta]\right)=-\varepsilon a^{\gamma\alpha}\sqrt{a_{(\gamma\gamma)}}\frac{\partial P}{\partial x^\alpha}+\frac{\partial}{\partial x^3}\left(\frac{\mu}{K}\frac{\partial u[\gamma]}{\partial x^3}\right)$$

$$\rho A^3_{\alpha\beta}u[\alpha]u[\beta]=-\frac{\partial P}{\partial x^3}$$

$$\rho DT=2\varepsilon\frac{u[\alpha]}{\sqrt{a_{(\alpha\alpha)}}}\frac{\partial P}{\partial x^\alpha}+\frac{1}{K}\frac{\partial}{\partial x^3}\left(\frac{\mu}{\sigma}\frac{\partial T}{\partial x^3}\right)+$$

$$\frac{2\mu}{K}\frac{a_{\alpha\beta}}{\sqrt{a_{(\alpha\alpha)}a_{(\beta\beta)}}}\frac{\partial u[\alpha]}{\partial x^3}\frac{\partial u[\beta]}{\partial x^3}$$

$$P=\rho T,\quad D\equiv\frac{u[\alpha]}{\sqrt{a_{(\alpha\alpha)}}}\frac{\partial}{\partial x^\alpha}+u[3]\frac{\partial}{\partial x^3}$$

$$T_0=\frac{V_\infty^2}{2c_p},\quad \mu_0=\mu(T_0),\quad \varepsilon=\frac{\gamma-1}{2\gamma},\quad \gamma=\frac{c_p}{c_v},\quad \sigma=\frac{\mu c_p}{\lambda}$$

$$\mathrm{Re}=\frac{\rho_\infty V_\infty R}{\mu_0},\quad \mathrm{Re}_w=\frac{\rho_w V_w R}{\mu_w},\quad G=\frac{\rho_w V_w}{\rho_\infty V_\infty}$$

$$\delta^2=\frac{\rho_w V_w{}^2}{\rho_\infty V_\infty^2},\quad K=\varepsilon\,\mathrm{Re}$$

$$A^\alpha_{\alpha\alpha}=\frac{a_{12}}{a}\left[\frac{\partial\sqrt{a_{\alpha\alpha}}}{\partial x^\beta}+\frac{a_{12}}{a_{(\alpha\alpha)}}\frac{\partial\sqrt{a_{(\alpha\alpha)}}}{\partial x^\alpha}-\frac{1}{\sqrt{a_{(\alpha\alpha)}}}\frac{\partial a_{12}}{\partial x^\alpha}\right]$$

$$A_{12}{}^\alpha=A_{21}{}^\alpha=\frac{1}{2a}\left[\sqrt{a_{11}a_{22}}\left(1+\frac{a_{12}{}^2}{a_{11}a_{22}}\right)\frac{\partial\sqrt{a_{\alpha\alpha}}}{\partial x^\beta}-2a_{12}\frac{\partial\sqrt{a_{\beta\beta}}}{\partial x^\alpha}\right]$$

$$A^\beta_{\alpha\alpha}=\frac{\sqrt{a_{(\beta\beta)}}}{a}\left[\frac{\partial a_{12}}{\partial x^\alpha}-\sqrt{a_{(\alpha\alpha)}}\frac{\partial\sqrt{a_{(\alpha\alpha)}}}{\partial x^\beta}-\frac{a_{12}}{\sqrt{a_{(\alpha\alpha)}}}\frac{\partial\sqrt{a_{(\alpha\alpha)}}}{\partial x^\alpha}\right](\beta\neq\alpha)$$

$$A^3_{\alpha\beta}=A^3_{\beta\alpha}=b_{\alpha\beta}\,/\sqrt{a_{(\alpha\alpha)}a_{(\beta\beta)}}$$

$$a=a_{11}a_{22}-a^2{}_{12},\qquad a_{33}=1$$

where summation is carried out over pairs of like indices, excluding in indices in parentheses which do not imply summation. Latin indices run through *1, 2, 3* and relate to space R^3, Greek indices assume the values *1, 2* and indicate association with the body surface contained in R^3; $V_\infty u[1]$, $V_\infty u[2]$, $\varepsilon V_\infty u[3]$ are physical components of the velocity vector, $\rho_\infty V_\infty{}^2 P$, $\varepsilon^{-1}\rho_\infty\ \rho$, $T_0 T$, $\mu_0\mu$, and λ are, respectively, the pressure, density, temperature, and the coefficients of viscosity and thermal conductivity, $c_p = \text{const}$ is the specific heat of gas; $a_{\alpha\beta}$, $b_{\alpha\beta}$ are components of symmetric covariant tensors that determine the first and second quadratic forms, respectively, of the body surface. All linear dimensions are normalized with respect to the characteristic linear dimension R, and the normal coordinate with respect to εR. Subscripts ∞ and w denote quantities in the oncoming stream and on the body surface, respectively. If an orthogonal coordinate system is selected on the body surface, $a_{12} = 0$.

Equations (1.1) are obtained from Navier — Stokes equations defined in the system of coordinates $\{x^i\}$ and physical components of vectors and tensors [13] in which ε and Re^{-1} approach zero and the product $K = \varepsilon \mathrm{Re}$ is of order unity. Terms with longitudinal pressure gradients are retained in Eqs. (1.1), since at high Reynolds numbers they are of considerable importance in the layer close to the body surface. At low Reynolds numbers they can be omitted. The system of Eqs. (1.1) thus defines the flow in a hypersonic viscous shock layer in a wide range of Reynolds numbers from moderately low to high. When $K \to \infty$ the degenerate equations (1.1) coincide with the equations that define in Newtonian theory the flow of inviscid gas [14 — 16]. It can be shown, as in [17], that in the presence of injection from the body surface under conditions $K \gg 1, \mathrm{Re}_w \gg 1, G \lesssim 1, \delta \ll 1$, system (1.1) provides an asymptotically correct definition of the flow of gas in the shock layer.

As the boundary conditions at the shock wave for $x^3 = x_s{}^3$ we use the modified Rankin — Hugoniot relations [1] in the thin layer approximation

$$u[\alpha] - u[\alpha]_\infty = \frac{\mu}{Ku[3]_\infty} \frac{\partial u[\alpha]}{\partial x^3}, \quad P = u^2[3]_\infty \tag{1.2}$$

$$\rho\left(u[3] - \frac{u[\alpha]}{\sqrt{a_{(\alpha\alpha)}}} \frac{\partial x_s{}^3}{\partial x^\alpha}\right) = u[3]_\infty$$

$$u[3]_\infty (H - H_\infty - u^2[3]_\infty) = \frac{\mu}{K\sigma} \frac{\partial T}{\partial x^3} + \frac{2\mu}{K}\left[u[\alpha]\frac{\partial u[\alpha]}{\partial x^3} + \frac{a_{12}}{\sqrt{a_{11}a_{22}}} \frac{\partial}{\partial x^3}(u[1]u[2])\right],$$

$$H = T + \frac{a_{\alpha\beta}}{\sqrt{a_{(\alpha\alpha)}a_{(\beta\beta)}}} u[\alpha]u[\beta]$$

At high Reynolds numbers $(K \to \infty)$ relations (1.2) convert to the conventional Rankin — Hugoniot formulas in hypersonic approximation.

We define the boundary conditions at the body surface in the absence of slip and temperature jump in the following dimensionless form:

$$x^3 = 0, \ u[\alpha] = 0, \ \rho u[3] = G(x^1, x^2), \ T = T_w(x^1, x^2) \tag{1.3}$$

The slip rate and temperature jump are quantities of the order of $\varepsilon^{1/2} K^{-1}$ [18].

2. Numerical solution of equations of a three-dimensional hypersonic viscous shock layer in the stagnation point neighborhood. For the numerical solution of the problem we substitute for the variables in Eqs. (1.1) the Dorodnitsyn's variables

$$\zeta = \frac{1}{\Delta}\int_0^{x^3} \rho\sqrt{a}\,dx^3, \quad x^\alpha = \xi^\alpha, \quad \Delta = \int_0^{x_s{}^3} \rho\sqrt{a}\,dx^3 \tag{2.1}$$

generally used in the theory of three-dimensional boundary layer (see, e.g., [19]), and introduce new variables using formulas

$$u[\alpha] = u^*_{(\alpha)}(\xi^1, \xi^2)\frac{\partial \varphi_\alpha}{\partial \zeta}, \quad T = T^*(\xi^1, \xi^2)\theta \tag{2.2}$$

$$\rho\sqrt{a}u[3] = -\frac{\partial\psi^*_{(\alpha)}(\xi^1,\xi^2)\varphi_\alpha}{\partial\xi^\alpha} - \Delta\varphi^*_{(\alpha)}\frac{\partial\varphi_2}{\partial\zeta}\frac{\partial\zeta}{\partial x^\alpha}$$
$$\psi_\alpha{}^* = \Delta\varphi_\alpha{}^* = \Delta\frac{u_\alpha{}^*}{\sqrt{a_{(\alpha\alpha)}}}, \quad l = \frac{\mu\rho a}{K\Delta^2}$$

with the equation of continuity indentically satisfied. Functions $u_\alpha{}^*(\xi^1, \xi^2)$, $T^*(\xi^1, \xi^2)$ will be defined below. The operator D in new variables is of the form

$$D = \varphi^*_{(\alpha)}\frac{\partial\varphi_\alpha}{\partial\zeta}\frac{\partial}{\partial\xi^\alpha} - \left[\varphi^*_{(\alpha)}\frac{\partial\varphi_\alpha}{\partial\xi^\alpha} + B_\alpha\varphi_\alpha\right]\frac{\partial}{\partial\zeta} \tag{2.3}$$
$$B_\alpha \equiv \frac{\partial\varphi^*_{(\alpha)}}{\partial\xi^\alpha} + \varphi^*_{(\alpha)}\frac{\partial\ln\Delta}{\partial\xi^\alpha}$$

Let us consider the flow over a smooth convex body whose surface S is specified in the Cartesian coordinate system in the form $y^3 = f(y^1, y^2)$. By selecting $x^\alpha = y^\alpha$ as the curvilinear coordinates on the surface of the body and the set of normals to surface S, as the coordinate lines x^3, we obtain

$$a_{\alpha\alpha} = 1 + q_\alpha{}^2, \; a_{12} = q_1q_2, \; b_{\alpha\beta} = -r_{\alpha\beta}/\sqrt{a} \tag{2.4}$$
$$a = 1 + q_1{}^2 + q_2{}^2, \quad q_\alpha \equiv \partial f/\partial x^\alpha, \quad r_{\alpha\beta} \equiv \partial^2 f/\partial x^\alpha\partial x^\beta$$

If the oncoming stream velocity vector coincides with the direction of the y^3-axis, the physical velocity components $u[i]_\infty$ in the thin layer approximation ($x^3 \approx 0$)are

$$u[\alpha]_\infty = \sqrt{a_{(\alpha\alpha)}}q_\alpha/a, \qquad u[3]_\infty = -1/\sqrt{a}$$

Let us, now, consider the flow over an elliptic paraboloid at zero angle of attack, whose surface we specify in the form $2y^3 = (y^1)^2 + k(y^2)^2$, where $k = R/R_1$; R and R_1 are the radii of principal curvatures at the parabolid stagnation point. Setting $u_\alpha{}^*(x^1, x^2) = u[\alpha]_\infty$, $T^*(x^1, x^2) = u^2[3]_\infty$ we resolve all singularities occurring in coefficients of equations for the stagnation point.

Using (1.1) and (2.1) — (2.4) we obtain for the three-dimensional flow of gas in the stagnation point neighborhood the following equations;

$$\frac{\partial}{\partial\zeta}\left(l\frac{\partial^2\varphi_\alpha}{\partial\zeta^2}\right) + (\varphi_1 + k\varphi_2)\frac{\partial^2\varphi_\alpha}{\partial\zeta^2} = d_\alpha\left(\frac{\partial\varphi_\alpha}{\partial\zeta}\right)^2 + \frac{\varepsilon}{\rho}P_\alpha \tag{2.5}$$
$$\frac{\partial}{\partial\zeta}\left(\frac{l}{\sigma}\frac{\partial\theta}{\partial\zeta}\right) + (\varphi_1 + k\varphi_2)\frac{\partial\theta}{\partial\zeta} = 0, \quad \frac{\partial P}{\partial\zeta} = 0$$
$$P = \rho\theta, \quad \frac{\partial P_\alpha}{\partial\zeta} = 2\Delta d_{(\alpha)}{}^2\left(\frac{\partial\varphi_\alpha}{\partial\zeta}\right)^2$$
$$P_\alpha \equiv \frac{1}{d_{(\alpha)}\xi^{(\alpha)}}\frac{\partial P}{\partial\xi^\alpha}, \quad d_1 = 1, \quad d_2 = k$$

The last of Eqs. (2.5) were obtained from the third of Eqs. (1.1) to which the operator $(d_{(\alpha)}\xi^{(\alpha)})^{-1}\,\partial/\partial\xi^\alpha$ was applied.

Boundary conditions (1.2) and (1.3) now assume the form

$$\zeta = 1, \; \varphi_1 + k\varphi_2 = 1/\Delta \tag{2.6}$$

$$\frac{\partial\varphi_\alpha}{\partial\zeta} + l\Delta\frac{\partial^2\varphi_\alpha}{\partial\zeta^2} = 1, \quad \theta + \frac{l\Delta}{\sigma}\frac{\partial\theta}{\partial\zeta} = 1, \quad P = 1, \quad P_\alpha = -2d_\alpha$$

$$\zeta = 0, \ \partial\varphi_\alpha / \partial\zeta = 0, \ \varphi_1 + k\varphi_2 = -G / \Delta, \ \theta = \theta_w \qquad (2.7)$$

Equations (2.5) with boundary conditions (2.6) and (2.7) were solved numerically with the following values of parameters: $k = 0,\ 0.1,\ 0.4$; 1.0; $\sigma = 0.71$; $\varepsilon = 0.1$; $\theta_w = 0.1,\ 0.5$; $-0.25 \leqslant G \leqslant 0.25$; $1 \leqslant \mathrm{Re} \leqslant 5\cdot 10^7$; $\mu = \theta^{1/2}$.

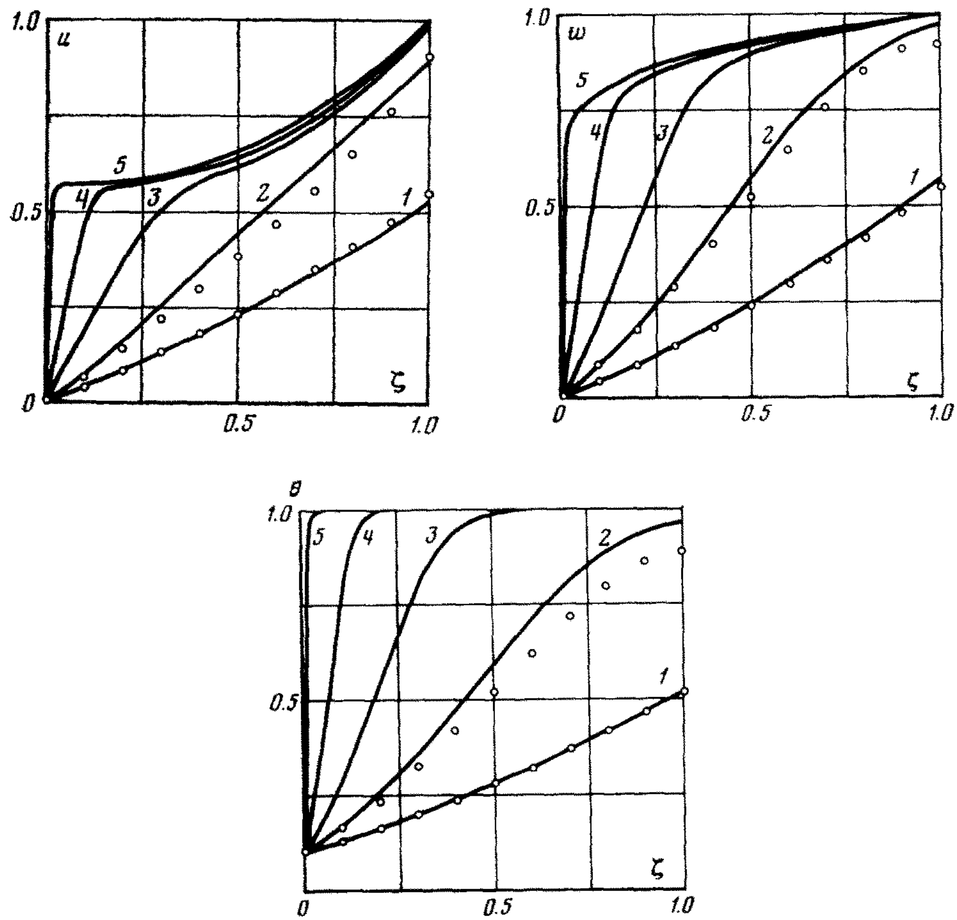

Fig. 1

The implicit finite-difference scheme [20] with approximation accuracy $O(\Delta\zeta^4)$ was used. To increase computation accuracy for high Reynolds numbers the computation grid was compressed near the surface of the body.

Some of the results of computations are shown in Figs. 1—6. Characteristic profiles of tangent velocity components $u = \partial\varphi_1 / \partial\zeta$, $w = \partial\varphi_2 / \partial\zeta$ and of temperature θ across the shock layer appear in Fig. 1 for $G = 0$, $k = 0.1$, $\theta_w = 0.1$ and $\mathrm{Re} = 5$ 50, $5\cdot 10^2$, $5\cdot 10^3$, $5\cdot 10^5$ (curves 1—5 respectively). The formation of a thin boundary layer at the body surface at high Reynolds numbers can be easily traced there. The profiles of u and θ for plane $(k = 0)$ and axisymmetric $(k = 1)$ flows were presented in [21].

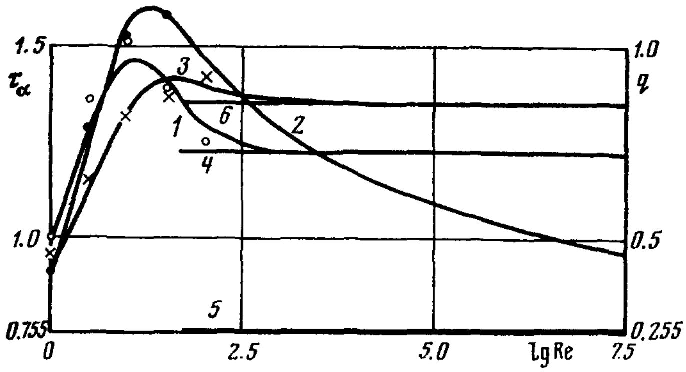

Fig. 2

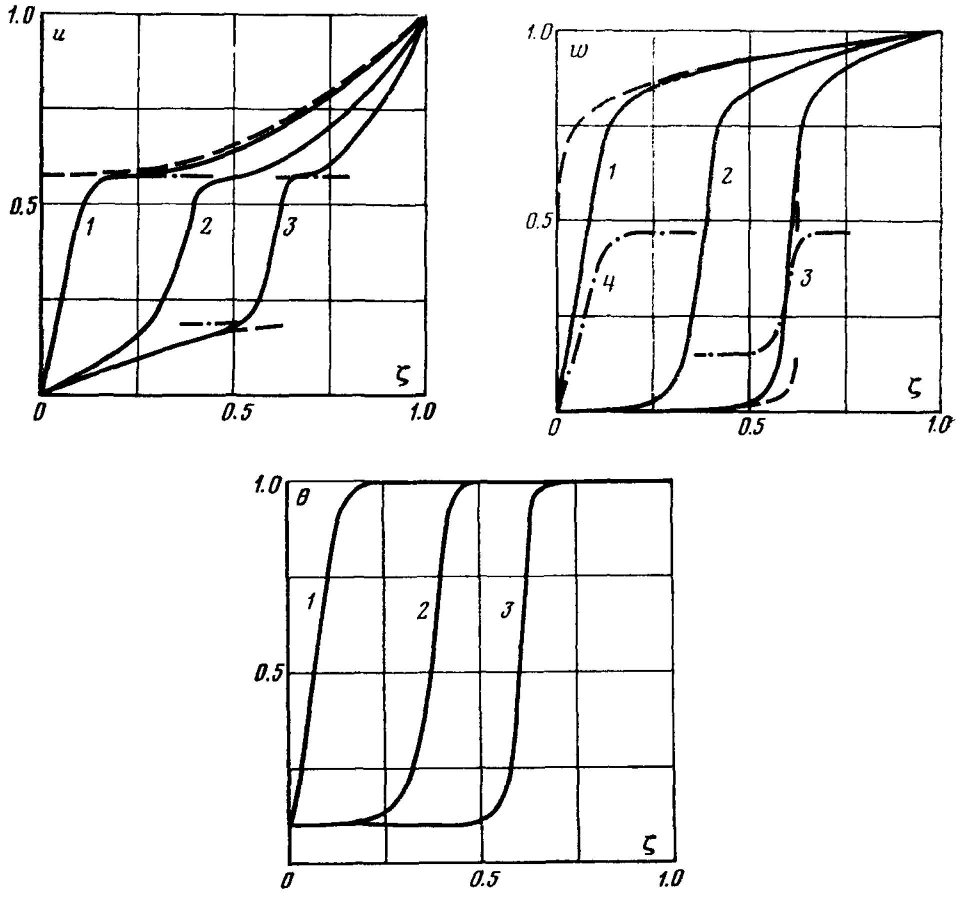

Fig. 3

The dependence of friction coefficients τ_1 and τ_2 (curves 1 and 2) and of heat exchange q (curve 3) on the Reynolds number, determined using the theory of viscous shock layer, is shown in Fig. 2, for $k = 0.1$ and $\theta_w = 0.1$. The straight lines 4—6 relate to respective coefficients of friction and heat exchange (straight lines 4 and 5 for τ_1 and τ_2, 6 for q) and were computed using the boundary layer theory (see Sect. 4). The formulas for τ_α and q are of the form

$$\tau'_{\xi\alpha\zeta} = \frac{\mu}{\rho_\infty V_\infty^2} \frac{\partial u[\alpha]}{\partial x^3} = u[\alpha]_\infty \frac{l\Delta}{\sqrt{a}} \frac{\partial^2 \varphi_\alpha}{\partial \zeta^2}\bigg|_{\zeta=0}, \quad \tau_\alpha = \frac{\tau'_{\xi\alpha\zeta}\sqrt{\mathrm{Re}}}{u[\alpha]_\infty} \qquad (2.8)$$

$$q' = \frac{\lambda}{\rho_\infty V_\infty^3} \frac{\partial T}{\partial x^3} = T^* \frac{l\Delta}{2\sigma\sqrt{a}} \frac{\partial \theta}{\partial \zeta}\bigg|_{\zeta=0}, \quad q = \frac{q'\sqrt{\mathrm{Re}}}{T^*}$$

The analysis of flow computation results obtained in [17, 21] shows that in a viscous shock layer the properties of gas flow in that layer are determined by the parameter $F = -G\sqrt{\mathrm{Re}}\,(1+k)^{-1/2}\varepsilon^{1/4}(-P_{1w})^{-1/4}$ generally used in the laminar boundary layer theory. The profiles of u, w and θ across the shock layer are shown in Fig. 3 by solid lines for Reynolds numbers $\mathrm{Re} = 5\cdot 10^3$, $k = 0.1$, $\theta_w = 0.1$ with $-F = 0$, 3.03, and 7.03 (curves 1—3 refer). It will be seen from Fig. 3 for $-F \geqslant 2$ that the boundary layer becomes separated from the body surface and forms the mixing region.

The dependence of coefficients of friction and heat exchange (normalized with respect to their values for $F = 0$) at the body surface on the injection parameter $-F$ is shown in Fig. 4 by solid lines (the following notation is used: $\tau_{\alpha f} = \tau_\alpha / (\tau_\alpha)_{F=0}$, $q_f = q/(q)_{F=0}$) for $k = 0.1$, $0 \leqslant G \leqslant 0.25$, $10 \leqslant \mathrm{Re} \leqslant 5\cdot 10^3$; curves 1 — 3 correspond to τ_{1f}, τ_{2f}, q_f, and the dash lines to the asymptotic solution of equations of the boundary layer with the strong injection [22]). Computations have

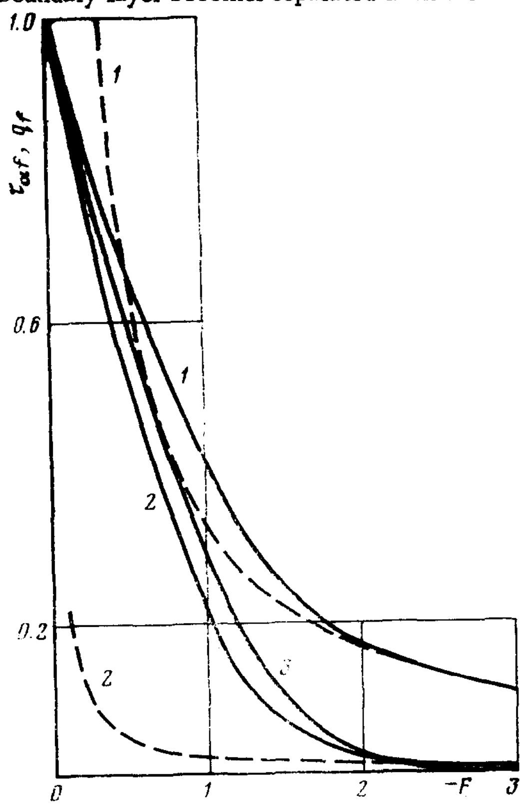

Fig. 4

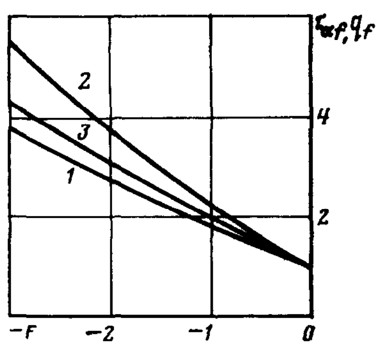

Fig. 5

shown that the points that correspond to various values of Re and G lie on a single curve; in this sense the obtained equations are universal.

The dependence of coefficients of friction and heat exchange normalized with respect to their values for $F = 0$ on the suction parameter F have also a universal character (see Fig. 5, where the notation is the same as in Fig. 4).

3. **Approximate analytic solution of equations of three-dimensional hypersonic viscous shock layer in the stagnation point neighborhood.** We introduce new unknown functions by formulas

$$v_\alpha = \frac{1}{c_\alpha}\frac{\partial \varphi_\alpha}{\partial \zeta}, \quad \vartheta = \frac{\theta - \theta_w}{\theta_s - \theta_w}, \quad c_\alpha = \left(\frac{\partial \varphi_\alpha}{\partial \zeta}\right)_s \tag{3.1}$$

where subscript s denotes parameters of gas immediately behind the shock wave. The boundary conditions for functions v_α and ϑ are

$$\zeta = 1,\ v_\alpha = \vartheta = 1; \quad \zeta = 0,\ v_\alpha = \vartheta = 0 \tag{3.2}$$

The system of Eqs. (2.5) with allowance for (3.1) and (3.2) can be reduced to the form

$$v_\alpha = 1 + F_\alpha(\zeta) + K_\alpha \int_1^\zeta l^{-1}\, d\zeta \tag{3.3}$$

$$\vartheta = 1 + F_3(\zeta) + K_3 \int_1^\zeta \sigma l^{-1}\, d\zeta$$

$$F_\alpha(\zeta) = \int_1^\zeta l^{-1} \int_0^\zeta \left[-(\varphi_1 + k\varphi_2)\frac{\partial v_\alpha}{\partial \zeta} + c_{(\alpha)} d_{(\alpha)} v_\alpha^2 + \frac{\varepsilon}{\rho c_{(\alpha)}} P_\alpha\right] d\zeta\, d\zeta$$

$$F_3(\zeta) = \int_1^\zeta \sigma l^{-1} \int_0^\zeta \left[-(\varphi_1 + k\varphi_2)\frac{\partial \vartheta}{\partial \zeta}\right] d\zeta\, d\zeta$$

$$K_\alpha = -[1 + F_\alpha(0)]\left(\int_1^0 l^{-1}\, d\zeta\right)^{-1}$$

$$K_3 = -[1 + F_3(0)]\left(\int_1^0 \sigma l^{-1}\, d\zeta\right)^{-1}$$

$$\varphi_1 + k\varphi_2 = c_{(\alpha)} d_\alpha \int_0^\zeta v_\alpha\, d\zeta - \frac{G}{\Delta}, \quad P = P_s = 1 \tag{3.4}$$

$$\rho^{-1} = \theta_w + (\theta_s - \theta_w)\,\vartheta$$

$$P_\alpha = 2\Delta c_{(\alpha)}^2 d_{(\alpha)}^2 \int_1^\zeta v_\alpha^2\, d\zeta - 2d_\alpha$$

and boundary conditions (2.6) in new variables become

$$\frac{1}{c_\alpha} = 1 + l\Delta\frac{\partial v_\alpha}{\partial \zeta}, \quad \frac{1 - \theta_s}{\theta_s - \theta_w} = \frac{l\Delta}{\sigma}\frac{\partial \vartheta}{\partial \zeta} \tag{3.5}$$

$$\frac{1+G}{\Delta} = \int_0^1 (c_1 v_1 + k c_2 v_2)\, d\zeta$$

The problem has been, thus, reduced to the simultaneous solution of Eqs. (3.3) and (3.5) for the variables v_α, ϑ, Δ, c_α, θ_s. The quantities P, P_α, ρ can be eliminated from (3.3) and (3.5) using expressions (3.4). To solve system (3.3), (3.4) we use the method of successive approximations [12] consisting of the following. Let the $(n-1)$-st approximation of functions $v_\alpha^{(n-1)}$ and $\vartheta^{(n-1)}$ be known. Substituting them into system (3.5) we determine Δ, c_α, θ_s. When $\Delta, c_\alpha, \theta_s$ are known, functions $v_\alpha^{(n)}$ and $\vartheta^{(n)}$, are determined by (3.3) after the substitution into their right-hand sides of $v_\alpha^{(n-1)}$, $\vartheta^{(n-1)}$. Repetition of this process yields subsequent approximations. The analysis of solutions of equations for a hypersonic viscous shock layer (Sect. 2, see also [2 —4, 8]) shows that the velocity and temperature profiles across the shock layer in terms of Dorodnitsyn's variables with parameter K of order unity, are close to linear. Hence, as the zero approximation we specify

$$v_\alpha^{(0)} = \vartheta^{(0)} = \zeta \tag{3.6}$$

Let us determine the analytic solution of the problem in the first approximation. Substituting (3.6) into (3.5) and specifying the viscosity coefficients as $\mu = \theta$, we obtain

$$\Delta = g + \sqrt{g^2 + 2gK^{-1}}, \quad g = (1+G)\,/\,(1+k) \tag{3.7}$$
$$c_\alpha = \Delta K\,/\,(1+\Delta K), \quad \theta_s = (\theta_w + \sigma\,\Delta K)\,/\,(1+\sigma\Delta K)$$

If the viscosity coefficient is proportional to θ^ω, $0 < \omega < 1$, the solution of system (3.5) can be obtained by determining for fixed σ, ω, θ_w the dependence on θ_s of the quantity

$$\begin{aligned} 2gK &= m\sigma^{-1}(\theta_s - \theta_w)\,(1-\theta_s)^{-1}\theta_s^{\,w-1} \\ m &= (\theta_s - \theta_w)\,[(1-\sigma)\,\theta_s + \sigma - \theta_w]^{-1} \end{aligned} \tag{3.8}$$

For the determination of c_α and Δ we have

$$c_\alpha = m, \quad \Delta = 2gm^{-1} \tag{3.9}$$

The substitution of (3.6) into (3.3) and (3.4) yields

$$F_n(\zeta) = \sum_{i=0}^{4} \frac{A_{ni}}{i+1}\left[\frac{a_n}{i+2}(\zeta^{i+2}-1) + \frac{b_n}{i+3}(\zeta^{i+3}-1)\right] \tag{3.10}$$

$$\begin{aligned}
&A_{\alpha 0} = \frac{G}{\Delta} + \frac{\varepsilon}{c_{(\alpha)}}\theta_w P_{\alpha 1}, \quad A_{\alpha 1} = \frac{\varepsilon}{c_{(\alpha)}}(\theta_s - \theta_w)\,P_{\alpha 1} \\
&A_{\alpha 2} = (-(1+k)\,/\,2 + d_{(\alpha)})\,c_\alpha, \quad A_{\alpha 3} = \varepsilon\theta_w P_{\alpha 2}\,/\,c_{(\alpha)} \\
&A_{\alpha 4} = \varepsilon\,(\theta_s - \theta_w)\,\,P_{\alpha 2}\,/\,c_{(\alpha)}, \quad A_{30} = G\,/\,\Delta \\
&A_{31} = A_{33} = A_{34} = 0, \quad A_{32} = -(1+k)\,c_\alpha\,/\,2 \\
&P_{\alpha 1} = -\,2d_{(\alpha)}\,(1 + \Delta d_{(\alpha)} c_\alpha^{\,2}\,/\,3), \quad P_{\alpha 2} = 2\Delta d_{(\alpha)}^{\,2}\,c_\alpha^{\,2}\,/\,3 \\
&a_\alpha = l_w^{-1}, \; b_\alpha = l_s^{-1} - l_w^{-1}, \; a_3 = \sigma a_\alpha, \; b_3 = \sigma b_\alpha
\end{aligned}$$

In the computation of integrals the quantity l was approximated thus:

$$l^{-1} = l_w^{-1} + (l_s^{-1} - l_w^{-1})\,\zeta$$

When Δ, c_α, θ_s, F_n, K_n are known, $\partial\varphi_\alpha / \partial\zeta$, θ are obtained from (3.3) and (3.1), and the coefficients of friction and heat exchange from formulas (2.8). Thus the problem is completely solved in the first approximation. Comparison of this solution (small circles) with the exact numerical solution is shown in Fig. 1 and in Fig. 2, (where small circles correspond to τ_1, dots to τ_2, and crosses to q).

The coefficients of friction and heat exchange were calculated by formulas (2.8) in which values of Δ calculated by the last of formulas (3.5) in whose right-hand side the values of $v_\alpha^{(1)}$ and $c_\alpha^{(1)}$, obtained in the first approximation, were substituted. The presented curves show that the approximate analytic solution is in good agreement with the numerical one for $0.1 \leqslant K \leqslant 5$. For higher values of parameter K that correspond to high Reynolds numbers, higher approximations are required. Similar results are obtained in the presence of injection.

4. **On the asymptotic solution of equations of three-dimensional hypersonic viscous shock layer in the stagnation point neighborhood at high Reynolds numbers.** We consider such Reynolds numbers at which the inequality $\varepsilon^2 \mathrm{Re} \gg 1$ is satisfied. In the case of high Reynolds numbers problem (2.5) – (2.7) is singular. Its asymptotic solution can be obtained using the method of external and internal expansions. The asymptotics of equations of the hypersonic viscous shock layer depends, then, on the injection parameter $-F$. When $-F \lesssim 1$ the shock layer can be divided into an inviscid layer and the boundary layer adjacent to the body surface. When $-F \gg 1$ a three-layer flow pattern obtains according to which effects of molecular transport may in the first approximation be neglected in the layers adjacent to the body surface and to the shock wave, while in the intermediate (mixing) region these effects play a predominant part. When solving the external problem the mixing layer can be replaced by a discontinuity surface with respective conditions on it [17, 21]. Some of the numerical solutions of the external problem are shown in Fig. 3 by dash lines (system (2.5) – (2.7) with $l \equiv 0$ and conditions $\varphi_1 + k\varphi_2 = 0$, $P_\alpha^+ = P_\alpha^-$ was solved at the contact discontinuity). The external problem can be solved in quadratures by setting in Eqs. (2.5) the longitudinal pressure gradient $P_\alpha(\zeta)$ equal $P_{\alpha w}$ which is determined by the Buseman – Hays formula (formula 3.6.35 in [15]) which is asymptotically correct for small values of parameter ε (see [4, 16, 17]). With this taken into account the solution of the external problem is of the form [16, 22]:
for the injection layer

$$u = \frac{s}{\sqrt{\rho_w}}, \quad w = \frac{\beta_2\sqrt{\rho_w}}{\sqrt{k}}\,\frac{(s+\beta_1)^\omega - (\beta_1 - s)^\omega}{(s+\beta_1)^\omega + (\beta_1 - s)^\omega} \tag{4.1}$$

$$t \equiv \frac{\Delta}{G}(\varphi_1 + k\varphi_2) = -\exp\int_0^s \frac{s + kw\rho_w^{-1/2}}{s^2 - \beta_1^2}\,ds, \quad \zeta^\circ = G\sqrt{\rho_w}\int_0^s \frac{t\,ds}{s^2 - \beta_1^2}$$

$$(0 \leqslant s \leqslant \beta_1), \quad \zeta_c^\circ = \zeta^\circ(\beta_1)$$

and for the shock layer $u = s$

$$w = \frac{\beta_2}{\sqrt{k}} \frac{(s+\beta_1)^\omega + C(s-\beta_1)^\omega}{(s+\beta_1)^\omega - C(s-\beta_1)^\omega}, \quad C = \frac{\sqrt{k}-\beta_2}{\sqrt{k}+\beta_2}\left(\frac{1+\beta_1}{1-\beta_1}\right)^\omega \qquad (4.2)$$

$$t \equiv \Delta(\varphi_1 + k\varphi_2) = \exp \int_{\beta_1}^{s} \frac{s+kw}{s^2-\beta_1^2}\, ds,$$

$$\zeta^\circ = \zeta_c^\circ + \int_{\beta_1}^{s} \frac{t\, ds}{s^2-\beta_1^2} \quad (\beta_1 \leqslant s \leqslant 1), \quad \Delta = \zeta^\circ(1)$$

$$(\beta_\alpha = \sqrt{-\varepsilon P_\alpha}, \ \omega = \sqrt{k}\beta_2/\beta_1, \ \zeta^\circ = \Delta\zeta)$$

where $(\zeta_c^\circ\Delta^{-1}$ is the thickness of the injected gas layer.

At small injection parameters $-F \lessapprox 1$ the internal problem in the usual approximation involves solution of boundary layer equations which in the stagnation point neighborhood are of the form of Eqs. (2.5) in which it is necessary to set $\Delta = 1$, $\partial P / \partial\zeta = \partial P_\alpha / \partial\zeta = 0$, $P_\alpha = P_{\alpha w}$, where $P_{\alpha w}$ is specified by the solution of the external problem.

Boundary conditions at the body surface are (2.7), and at infinity are of the usual form

$$\zeta \to \infty, \ \partial\varphi_\alpha / \partial\zeta \to (-\varepsilon P_{\alpha w} / d_{(\alpha)})^{1/2}, \ \theta \to 1 \qquad (4.3)$$

The system of equations of the boundary layer with boundary conditions (2.7) and (4.3) in the case of injection or suction were solved numerically using the same method and the same grids, as for the system of shock layer equations.

Comparison of coefficients of friction and heat exchange are shown in Fig. 2, where the straight lines are solutions of boundary layer equations. It will be seen that the difference in the friction coefficient τ_2 in the direction coinciding with the plane of the body greatest radius of curvature can be considerable even for high Reynolds numbers. The difference in coefficients τ_1 and q is considerably smaller. This difference is explained as follows.

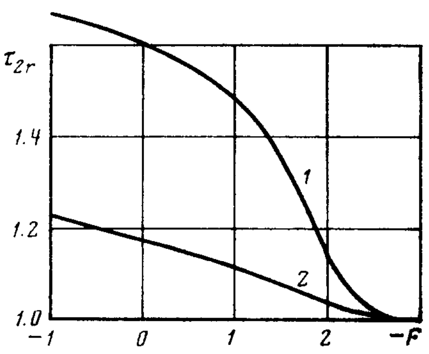

Fig. 6

The analysis of solution of the external problem (4.2) shows that a thin layer of strong vorticity is formed close to the surface of a body elongated in the transverse direction (small k) in the flow past it. One of the tangent components of the velocity vector has an infinite derivative with respect to the transverse variable on the body surface (see Fig. 3, where lines 4 and 1 have been calculated for the same values of parameters). The other velocity component and the temperature have zero derivatives with respect to the transverse coordinate when $0 < k < 1$. For a correct determination of asymptotics of shock layer equations for high Reynolds numbers it is, thus, necessary to take into account the effect of vorticity which is usually omitted in the conventional formulation of the first approximation of the problem in the theory of the boundary layer.

Note that in the case of plane flow the derivative of the tangent velocity component with respect to the transverse coordinate vanishes, while in the case of an axisymmetric flow it is finite [21], which at high Reynolds numbers results in the reduction of the external flow vorticity effect on the coefficients of friction and heat exchange at the body surface.

The ratio of friction coefficients τ_{2r} on the body surface in the direction of ξ^2 plotted in Fig. 6 obtained by solving equations of the shock and boundary layers (curves *1* and *2* correspond to $k = 0.1$ and 0.4), show that as the suction parameter is increased, this ratio increases because then the principal terms in the boundary layer equations are associated with molecular transport effects [23]. With increased injection the ratio of friction coefficients decreases, and beginning at an injection parameter $-F \geqslant 2.5$ that ratio remains close to unity. This is explained by that the basic factor in the coefficient of friction at the body surface at high injection parameters is then the pressure gradient in the longitudinal direction [17, 22]. In the transverse direction of the injected gas layer that parameter varies only little. It can be shown that for $-F \gg 1$ and $\delta \ll 1$ the ratio of friction coefficients in the shock and boundary layers is of order

$$\tau_{\alpha r} \to \frac{P_{\alpha\,(F\neq 0)}}{P_{\alpha\,(F=0)}} = 1 + O(\delta)$$

For $-F \gg 1$ solution of the internal problem consists of solving equations of the boundary layer with conditions (4.3) and conditions [17]

$$\zeta \to -\infty,\ \partial\varphi_\alpha / \partial\zeta \to [-\varepsilon P_{\alpha w} / \rho_w d_{(\alpha)}]^{1/2},\ \theta \to \theta_w$$
$$\zeta = \zeta_c, \quad \varphi_1 + k\varphi_2 = 0$$

Some results of the solution of this problem are shown in Fig. 3 by dash-dot lines from which it can be seen that the difference in profiles of u and θ calculated by equations of the viscous shock layer and by equations of the mixing layer are insignificant in the viscous part of the shock layer, while in the profiles of w it is considerably greater. In this case a layer of high vorticity is formed near the surface of contact discontinuity.

The author thanks G. A. Tirskii and G. G. Chernyi for discussing this paper.

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Translated by J. J. D.